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Ternary Codes and Vertex Operator Algebras[1]

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1. INTRODUCTION

Vertex operator algebras have been studied from a wide variety of view point. This implies rich properties of vertex operator algebras. Recently investigation of vertex operator algebras as modules for their subalgebras isomorphic to a tensor product $\otimes_{i=1}^{n} L(c_i, 0)$ of Virasoro vertex operator algebras was initiated by Dong *et al*. [DMZ]. Along this line Miyamoto [M2] constructed a series of vertex operator algebras by combining the minimal vertex operator super algebra $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$ with even binary codes.

In this article we construct vertex operator algebras associated with self orthogonal ternary codes. We begin with a lattice $L = \sqrt{2}\,(A_2\text{-lattice})$. It is known [DLMN] that the vertex operator algebra V_L contains a subalgebra T isomorphic to $L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes L(\frac{4}{5}, 0)$. Inspecting the action of T we obtain a vertex operator algebra isomorphic to $L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$ and two of its modules, both of which is isomorphic to $L(\frac{4}{5}, \frac{2}{3})$. Combining

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them with a self orthogonal ternary code, we construct a vertex operator algebra. We also prove the rationality of $L(\frac{4}{5},0) \oplus L(\frac{4}{5},3)$ and determine its Zhu algebra [Z1, Z2].

The irreducible modules for the vertex operator algebra $L(\frac{4}{5},0) \oplus L(\frac{4}{5},3)$ have been already known in physics literature [FZa, Za]. In this article we construct them explicitly in $V_{L^\perp}$, where $L^\perp$ is the dual lattice of L.

Our notation is standard ([FHL, FLM]). If (M, Y^M) is a module for a vertex operator algebra (V, Y) and $v \in V$, we shall write $Y^M(v,z) = \sum_{n\in\mathbb{Z}} v_n^M z^{-n-1}$ with $v_n^M \in \operatorname{End} M$ to distinguish it from $Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ with $v_n \in \operatorname{End} V$.

2. LATTICE Γ_D

Let $\{\alpha_1, \alpha_2\}$ be a set of fundamental roots of type A_2 with an inner product $\langle \cdot,\cdot \rangle$ such that $\langle \alpha_i, \alpha_i \rangle = 2$ and $\langle \alpha_1, \alpha_2 \rangle = -1$. Set $\alpha_3 = \alpha_1 + \alpha_2$. Let $L = \mathbb{Z}\alpha_1^* + \mathbb{Z}\alpha_2^*$, where $\alpha_i^* = \sqrt{2}\,\alpha_i$. For simplicity, we also write $x = \alpha_1^*$ and $y = \alpha_2^*$. The dual lattice $L^\perp$ of L has the dual basis $\{\frac{2x+y}{6}, \frac{x+2y}{6}\}$ and L has 12 cosets in $L^\perp$. Among them we choose

$$L^0 = L, \qquad L^1 = \frac{-x+y}{3} + L, \qquad L^2 = \frac{x-y}{3} + L.$$

In fact, $2L^\perp = L^0 \cup L^1 \cup L^2$ and the quotient group $2L^\perp/L = \{L^0, L^1, L^2\}$ is of order 3.

Let D be a ternary code of length n, that is, a subspace of an n dimensional vector space over $GF(3) = \{0,1,2\}$. For each codeword $\delta = (d_1, \dots, d_n)$, we assign a subset L_δ of an orthogonal sum of n copies of $L^\perp$

$$L_\delta = L^{d_1} \oplus \cdots \oplus L^{d_n} \subset \left(L^\perp\right)^n.$$

Then since $L^i + L^j = L^{i+j}$ for $i, j \in \{0,1,2\}$, where the superscript $i+j$ is considered to be modulo 3, the union $\Gamma_D = \bigcup_{\delta\in D} L_\delta$ of all L_δ; $\delta \in D$ is a sublattice of $(L^\perp)^n$.

Note that $\langle \alpha, \beta \rangle \in \frac{4}{3}dg + 2\mathbb{Z}$ if $\alpha \in L^d$ and $\beta \in L^g$ for $d, g \in \{0,1,2\}$. Let $\delta = (d_1, \dots, d_n)$ and $\gamma = (g_1, \dots, g_n)$ be two codewords of D and choose $\alpha^i \in L^{d_i}$, $\beta^i \in L^{g_i}$. Denote by $\tilde{\alpha}$ and $\tilde{\beta}$ the elements $(\alpha^1, \dots, \alpha^n)$ of L_δ and $(\beta^1, \dots, \beta^n)$ of L_γ, respectively. Then

$$\langle \tilde{\alpha}, \tilde{\beta} \rangle = \sum_{i=1}^n \langle \alpha^i, \beta^i \rangle \in \frac{4}{3}\delta\cdot\gamma + 2\mathbb{Z},$$

where $\delta\cdot\gamma = d_1 g_1 + \cdots + d_n g_n$.

If D is a self orthogonal ternary code of length n, then Γ_D is a doubly even lattice of rank $2n$, that is, $\langle \alpha, \alpha \rangle \in 4\mathbb{Z}$ for $\alpha \in \Gamma_D$. For example, if $n = 3$ and $D = \{(0,0,0), \pm(1,1,1)\}$, then $\Gamma_D = \sqrt{2}(E_6\text{-lattice})$. If $n = 4$ and letting

$$D = \mathscr{C}_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

be the $[4,2,3]$ ternary tetracode, then $\Gamma_D = \sqrt{2}(E_8\text{-lattice})$.

3. VERTEX OPERATOR $Y(\cdot, z)$

We shall consider the Fock space $V_{L^\perp}$ associated with the lattice $L^\perp$ and the vertex operator $Y(v,z) = \sum_{n\in\mathbb{Q}} v_n z^{-n-1} \in (\operatorname{End} V_{L^\perp})\{z\}$ for $v \in V_{L^\perp}$ as in [D, Section 2, DL, Chapter 3, FLM, Chapters 4, 7, 8]. We also consider the subspaces $V^0 = V_L$, $V^1 = V_{L^1}$, and $V^2 = V_{L^2}$ of $V_{L^\perp}$ associated with the cosets $L^0 = L$, L^1, and L^2 of L in $L^\perp$. Here, instead of a twisted group algebra $\mathbb{C}\{L^\perp\}$, we use the group algebra $\mathbb{C}[L^\perp]$ with basis e^α; $\alpha \in L^\perp$ and multiplication $e^\alpha e^\beta = e^{\alpha+\beta}$.

Since $\langle L, 2L^\perp \rangle \subset 2\mathbb{Z}$, $\langle L^i, L^j \rangle \subset \frac{1}{3}\mathbb{Z}$, and $L^i + L^j = L^{i+j}$, we have

$$Y(u,z)v = \begin{cases} \displaystyle\sum_{n\in\mathbb{Z}} u_n v z^{-n-1} & \text{with } u_n v \in V^i \\ \qquad \text{for } u \in V^0,\ v \in V^i, & \\ \displaystyle\sum_{n\in\frac{1}{3}\mathbb{Z}} u_n v z^{-n-1} & \text{with } u_n v \in V^{i+j} \\ \qquad \text{for } u \in V^i,\ v \in V^j. & \end{cases} \tag{3.1}$$

The Jacobi identity [DL, (5.11)] gives a useful formula

$$u_m v_n w - v_n u_m w = \sum_{i=0}^{\infty} \binom{m}{i} (u_i v)_{m+n-i} w \tag{3.2}$$

for $u \in V^0$, $v, w \in V_{2L^\perp}$, $m \in \mathbb{Z}$, and $n \in \frac{1}{3}\mathbb{Z}$.

As a summary, (V^0, Y) is a vertex operator algebra whose Virasoro element is

$$\omega = \tfrac{1}{6}\left(\alpha_1(-1)^2 + \alpha_2(-1)^2 + \alpha_3(-1)^2\right),$$

(V^1, Y) and (V^2, Y) are V^0-modules, and

$$\begin{aligned} Y(\cdot, z)\colon V^i &\to \operatorname{Hom}(V^j, V^{i+j})\{z\} \\ v &\mapsto Y(v,z)|_{V^j} = \sum_{n\in\frac{1}{3}\mathbb{Z}} v_n|_{V^j} z^{-n-1} \end{aligned} \tag{3.3}$$

is an intertwining operator of type

$$\begin{pmatrix} & V^{i+j} & \\ V^i & & V^j \end{pmatrix}.$$

Let D be a self orthogonal ternary code of length n. For each codeword $\delta = (d_1, \ldots, d_n)$ we assign a tensor product $V_\delta = V^{d_1} \otimes \cdots \otimes V^{d_n}$ of vector spaces. For $v^i \in V^{d_i}$, define the tensor product vertex operator

$$Y(v^1 \otimes \cdots \otimes v^n, z) = Y(v^1, z) \otimes \cdots \otimes Y(v^n, z)$$

as in [DL, FHL]. Set $V_D = \bigoplus_{\delta \in D} V_\delta$. Then (V_D, Y) is a vertex operator algebra. In fact, it is identical to the vertex operator algebra V_{Γ_D} associated with the lattice Γ_D of Section 2.

4. VERTEX OPERATOR ALGEBRA M_D

By [DLMN], the Virasoro element ω of V^0 can be written as a sum of three mutually orthogonal conformal vectors

$$\begin{aligned}
\omega^1 &= \tfrac{1}{8}\alpha_1(-1)^2 - \tfrac{1}{4}x_{\alpha_1}, \\
\omega^2 &= \tfrac{1}{40}\left(-\alpha_1(-1)^2 + 4\alpha_2(-1)^2 + 4\alpha_3(-1)^2\right) \\
&\quad - \tfrac{1}{20}\left(-x_{\alpha_1} + 4x_{\alpha_2} + 4x_{\alpha_3}\right), \\
\omega^3 &= \tfrac{1}{15}\left(\alpha_1(-1)^2 + \alpha_2(-1)^2 + \alpha_3(-1)^2\right) + \tfrac{1}{5}(x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3}),
\end{aligned}$$

where $x_{\alpha_i} = e^{\alpha_i^*} + e^{-\alpha_i^*}$. The central charge $c(\omega^i)$ of the conformal vector ω^i is $c(\omega^1) = 1/2$, $c(\omega^2) = 7/10$, $c(\omega^3) = 4/5$, and the central charge of ω is $c(\omega) = 2$. Each conformal vector ω^i generates a Virasoro vertex operator algebra [M1] $\mathrm{Vir}(\omega^i) \cong L(c(\omega^i), 0)$ and V^0 contains a subalgebra

$$T = \mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2) \otimes \mathrm{Vir}(\omega^3) \cong L\left(\tfrac{1}{2}, 0\right) \otimes L\left(\tfrac{7}{10}, 0\right) \otimes L\left(\tfrac{4}{5}, 0\right).$$

As T-modules, V^i's are completely reducible and each irreducible summand is of the form [DMZ, W]

$$L\left(\tfrac{1}{2}, h_1\right) \otimes L\left(\tfrac{7}{10}, h_2\right) \otimes L\left(\tfrac{4}{5}, h_3\right),$$

where

$$\begin{aligned} &h_1 \in \left\{0, \tfrac{1}{16}, \tfrac{1}{2}\right\}, \\ &h_2 \in \left\{0, \tfrac{3}{80}, \tfrac{1}{10}, \tfrac{7}{16}, \tfrac{3}{5}, \tfrac{3}{2}\right\}, \\ &h_3 \in \left\{0, \tfrac{1}{40}, \tfrac{1}{15}, \tfrac{1}{8}, \tfrac{2}{5}, \tfrac{21}{40}, \tfrac{2}{3}, \tfrac{7}{5}, \tfrac{13}{8}, 3\right\}. \end{aligned} \tag{4.1}$$

Since V^0 is generated by the homogeneous subspaces $(V^0)_{(1)}$ and $(V^0)_{(2)}$, calculating the eigenvalues of the action of ω_1^i, $i = 1, 2, 3$ on these two homogeneous subspaces and using the fusion rules [W] and the dimension of $L(c, h)_{(n)}$ [FF, R], we have

LEMMA 4.1. *V^0 is a direct sum of the following irreducible T-submodules, each of which is of multiplicity one*:

$$\begin{gathered} L\left(\tfrac{1}{2}, 0\right) \otimes L\left(\tfrac{7}{10}, 0\right) \otimes L\left(\tfrac{4}{5}, 0\right), \\ L\left(\tfrac{1}{2}, 0\right) \otimes L\left(\tfrac{7}{10}, \tfrac{3}{5}\right) \otimes L\left(\tfrac{4}{5}, \tfrac{2}{5}\right), \\ L\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \otimes L\left(\tfrac{7}{10}, \tfrac{1}{10}\right) \otimes L\left(\tfrac{4}{5}, \tfrac{2}{5}\right), \\ L\left(\tfrac{1}{2}, 0\right) \otimes L\left(\tfrac{7}{10}, \tfrac{3}{5}\right) \otimes L\left(\tfrac{4}{5}, \tfrac{7}{5}\right), \\ L\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \otimes L\left(\tfrac{7}{10}, \tfrac{1}{10}\right) \otimes L\left(\tfrac{4}{5}, \tfrac{7}{5}\right), \\ L\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \otimes L\left(\tfrac{7}{10}, \tfrac{3}{2}\right) \otimes L\left(\tfrac{4}{5}, 0\right), \\ L\left(\tfrac{1}{2}, 0\right) \otimes L\left(\tfrac{7}{10}, 0\right) \otimes L\left(\tfrac{4}{5}, 3\right), \\ L\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \otimes L\left(\tfrac{7}{10}, \tfrac{3}{2}\right) \otimes L\left(\tfrac{4}{5}, 3\right). \end{gathered}$$

Similarly we have

LEMMA 4.2. *V^1 has the following irreducible T-submodules as direct summands, each of which is of multiplicity one*:

$$\begin{gathered} L\left(\tfrac{1}{2}, 0\right) \otimes L\left(\tfrac{7}{10}, 0\right) \otimes L\left(\tfrac{4}{5}, \tfrac{2}{3}\right), \\ L\left(\tfrac{1}{2}, 0\right) \otimes L\left(\tfrac{7}{10}, \tfrac{3}{5}\right) \otimes L\left(\tfrac{4}{5}, \tfrac{1}{15}\right), \\ L\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \otimes L\left(\tfrac{7}{10}, \tfrac{1}{10}\right) \otimes L\left(\tfrac{4}{5}, \tfrac{1}{15}\right). \end{gathered}$$

The minimal weight of any other direct summand is greater than $\frac{2}{3}$. The decomposition of V^2 into a direct sum of irreducible T-submodules is the same as that of V^1.

THEOREM 4.3. *Set $M^i = \{v \in V^i | \omega_1^1 v = \omega_1^2 v = 0\}$. Then*

(1) $M^0 = \mathbf{1}_{L(1/2,0)} \otimes \mathbf{1}_{L(7/10,0)} \otimes (L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3))$. *The weight* 2 *subspace of M^0 is spanned by $\mathbf{v}^0 = \omega^3$.*

(2) $M^1 = \mathbf{1}_{L(1/2,0)} \otimes \mathbf{1}_{L(7/10,0)} \otimes L(\frac{4}{5}, \frac{2}{3})$. *The weight* $\frac{2}{3}$ *subspace of* M^1 *is spanned by*

$$\mathbf{v}^1 = e^{(-x+y)/3} + e^{(2x+y)/3} + e^{(-x-2y)/3}.$$

(3) $M^2 = \mathbf{1}_{L(1/2,0)} \otimes \mathbf{1}_{L(7/10,0)} \otimes L(\frac{4}{5}, \frac{2}{3})$. *The weight* $\frac{2}{3}$ *subspace of* M^2 *is spanned by*

$$\mathbf{v}^2 = e^{(x-y)/3} + e^{(x+2y)/3} + e^{(-2x-y)/3}.$$

Here, $\mathbf{1}_{L(c,0)}$ *denotes the vacuum of* $L(c,0)$. *Moreover,* $u_n v \in M^{i+j}$ *for* $u \in M^i$ *and* $v \in M^j$. *Thus* (M^0, Y) *is a subalgebra of* (V^0, Y) *with the Virasoro element* ω^3, M^1 *and* M^2 *are* M^0*-modules, and the restriction of* (3.3) *to* M^i *is an intertwining operator of type*

$$\begin{pmatrix} & M^{i+j} & \\ M^i & & M^j \end{pmatrix}.$$

Proof. (1) follows from Lemma 4.1. Similarly, M^i is a direct sum of subspaces of the form $\mathbf{1}_{L(1/2,0)} \otimes \mathbf{1}_{L(7/10,0)} \otimes L(\frac{4}{5}, h_3)$ with h_3 as in (4.1). The weights of V^1 and V^2 are in $\frac{2}{3} + \mathbb{Z}_{\geq 0}$, and so (2) and (3) hold. By (3.1) (see also (3.2) and [M1, Proposition 4.9]), the last assertion holds. ∎

Exchange of the fundamental roots α_1 and α_2 induces an automorphism of the lattice L and so it induces a linear automorphism ρ of $V_{L^\perp}$ of order two. Since $\rho Y(u,z)v = Y(\rho u, z)\rho v$ for $u, v \in V_{L^\perp}$ and since $\rho(\omega^1 + \omega^2) = \omega^1 + \omega^2$ and $\rho\omega^3 = \omega^3$, it follows that $\rho(M^0) = M^0$ and $\rho(M^1) = M^2$. Also ρ acts on the $L(\frac{4}{5}, 0)$-part of M^0 as the identity.

Now consider two homogeneous elements of weight 3:

$$\begin{aligned} v_s = {} & \frac{\sqrt{2}}{3}\big(\alpha_1(-1)\alpha_1(-2) + \alpha_2(-1)\alpha_2(-2) + \alpha_3(-1)\alpha_3(-2)\big) \\ & + \alpha_1(-1)(e^{\alpha_1^*} - e^{-\alpha_1^*}) + \alpha_2(-1)(e^{\alpha_2^*} - e^{-\alpha_2^*}) \\ & + \alpha_3(-1)(e^{\alpha_3^*} - e^{-\alpha_3^*}), \end{aligned}$$

and

$$\begin{aligned} q = {} & \tfrac{1}{9}\big(\alpha_1(-1) - \alpha_2(-1)\big)\big(\alpha_1(-1) + 2\alpha_2(-1)\big)\big(2\alpha_1(-1) + \alpha_2(-1)\big) \\ & + \tfrac{1}{2}\big(\alpha_1(-1) + 2\alpha_2(-1)\big)(e^{\alpha_1^*} + e^{-\alpha_1^*}) \\ & - \tfrac{1}{2}\big(2\alpha_1(-1) + \alpha_2(-1)\big)(e^{\alpha_2^*} + e^{-\alpha_2^*}) \\ & + \tfrac{1}{2}\big(\alpha_1(-1) - \alpha_2(-1)\big)(e^{\alpha_3^*} + e^{-\alpha_3^*}). \end{aligned}$$

We have

$$\left(\omega_1^1+\omega_1^2\right)v_s=\left(\omega_1^1+\omega_1^2\right)q=0, \qquad \omega_2^3 v_s\neq 0, \qquad \omega_2^3 q=0.$$

Hence v_s and q form a basis of $M^0_{(3)}$ and q is a highest weight vector in the $L(\frac{4}{5},3)$-part of M^0. Since $\rho v_s=v_s$ and $\rho q=-q$, v_s is contained in the $L(\frac{4}{5},0)$-part of M^0 and ρ acts as -1 on the $L(\frac{4}{5},3)$-part of M^0. By a direct calculation we have

$$q_2\mathbf{v}^1=\frac{13\sqrt{2}}{9}\mathbf{v}^1, \qquad q_2\mathbf{v}^2=-\frac{13\sqrt{2}}{9}\mathbf{v}^2, \qquad \text{and} \qquad q_5 q=13\mathbf{1}.$$

Let D be a self orthogonal ternary code of length n. For each codeword $\delta=(d_1,\dots,d_n)$ we assign the tensor product $M_\delta=M^{d_1}\otimes\cdots\otimes M^{d_n}$ and set $M_D=\bigoplus_{\delta\in D}M_\delta$. Then by Theorem 4.3, M_D is a subalgebra of V_D. If $\delta=(0,\dots,0)$, M_δ contains an element u^i whose ith entry is $\mathbf{v}^0$ and the other entries are the vacuum $\mathbf{1}$ of M^0:

$$u^i=\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes\mathbf{v}^0\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}.$$

Set $\hat{\omega}=u^1+u^2+\cdots+u^n$. Then

THEOREM 4.4. *M_D is a vertex operator algebra with the vacuum $\mathbf{1}\otimes\cdots\otimes\mathbf{1}$ and the Virasoro element $\hat{\omega}$. Moreover, $(M_D)_{(0)}$ is of dimension one and $(M_D)_{(1)}=0$.*

Clearly the automorphism group $\operatorname{Aut}D$ of the code D induces a subgroup of $\operatorname{Aut}M_D$. Let $\varepsilon=\exp(2\pi\sqrt{-1}/3)$ and take $(j_1,\dots,j_n)\in GF(3)^n$. Then for $v^i\in M^{d_i}$,

$$v^1\otimes\cdots\otimes v^n\mapsto\varepsilon^{j_1d_1+\cdots+j_nd_n}v^1\otimes\cdots\otimes v^n$$

defines an automorphism of M_D called a coordinate automorphism [M2, Section 5]. This automorphism is trivial if and only if $(j_1,\dots,j_n)$ is orthogonal to D. Thus the coordinate automorphisms form a subgroup of $\operatorname{Aut}M_D$ isomorphic to $GF(3)^n/D^\perp$.

Since D is self orthogonal and $(M_D)_{(2)}$ comes from M_δ with δ of weight 0 or 3, we may assume that D is an orthogonal sum of the examples in Section 2 of length 3 or 4 and the zero code. In the case of ternary tetracode we have

PROPOSITION 4.5. *Let D be the $[4,2,3]$ ternary tetracode. Then $\operatorname{Aut}M_D\cong 3^2.2S_4$, where $2S_4$ is $\operatorname{Aut}D$ and 3^2 is the coordinate automorphisms.*

Proof. In this case the Griess algebra $\mathscr{G}=(M_D)_{(2)}$ has a basis u^i; $1\le i\le 4$ and x^δ; $\delta=(d_1,d_2,d_3,d_4)\neq 0$. Here $x^\delta=x^{d_1}\otimes\cdots\otimes x^{d_4}$ with $x^0=\mathbf{1}$, $x^1=\mathbf{v}^1$, and $x^2=\mathbf{v}^2$. The product $\times$ defined by $u\times v=u_1v$ and the inner product $\langle\,\cdot\,,\cdot\,\rangle$ defined by $\langle u,v\rangle\mathbf{1}=u_3v$ ([FLM, Section 8.9]) are as follows:

$$u^i\times u^j=2\delta_{ij}u^i,$$

$$u^i\times x^\delta=\begin{cases}\frac{2}{3}x^\delta & \text{if } d_i\neq 0,\\ 0 & \text{if } d_i=0,\end{cases}$$

$$x^\delta\times x^\gamma=\begin{cases}8x^{-\delta} & \text{if } \delta=\gamma,\\ 45\sum\limits_{d_i\neq 0}u^i & \text{if } \delta=-\gamma,\\ 6x^{\delta+\gamma} & \text{if } \delta\neq\pm\gamma,\end{cases}$$

$$\langle u^i,u^j\rangle=\tfrac{2}{5}\delta_{ij},$$

$$\langle u^i,x^\delta\rangle=0,$$

$$\langle x^\delta,x^\gamma\rangle=\begin{cases}27 & \text{if } \delta=-\gamma,\\ 0 & \text{if } \delta\neq-\gamma.\end{cases}$$

For $u,v\in\mathscr{G}$, let $\beta(u,v)$ be the trace of $\tau_u\tau_v\in\operatorname{End}\mathscr{G}$, where τ_u: $w\mapsto u\times w$. Then we have

$$\beta(u^i,u^j)=\begin{cases}\frac{20}{3} & \text{if } i=j,\\ \frac{16}{9} & \text{if } i\neq j,\end{cases}$$

$$\beta(u^i,x^\delta)=0,$$

$$\beta(x^\delta,x^\gamma)=\begin{cases}460 & \text{if } \delta=-\gamma,\\ 0 & \text{if } \delta\neq-\gamma.\end{cases}$$

The automorphism group $\operatorname{Aut}\mathscr{G}$ of the Griess algebra $\mathscr{G}$ consists of linear automorphisms g which preserve the product and the inner product, $gu\times gv=g(u\times v)$ and $\langle gu,gv\rangle=\langle u,v\rangle$. Note that $\beta(gu,gv)=\beta(u,v)$ for $g\in\operatorname{Aut}\mathscr{G}$.

For convenience, set $w^1=u^1+u^2+u^3+u^4$, $w^2=u^1-u^2$, $w^3=u^1+u^2-2u^3$, and $w^4=u^1+u^2+u^3-3u^4$. Now $30\langle\,\cdot\,,\cdot\,\rangle-\beta(\cdot,\cdot)$ is a bilinear form on $\mathscr{G}$ invariant under $\operatorname{Aut}\mathscr{G}$. Hence its kernel $\operatorname{span}\{w^1\}$ is invariant under $\operatorname{Aut}\mathscr{G}$. Similarly the kernel $\operatorname{span}\{w^2,w^3,w^4\}$ of $110\langle\,\cdot\,,\cdot\,\rangle-9\beta(\cdot,\cdot)$ and the kernel $\operatorname{span}\{x^\delta;0\neq\delta\in D\}$ of $460\langle\,\cdot\,,\cdot\,\rangle-27\beta(\cdot,\cdot)$ are invariant under $\operatorname{Aut}\mathscr{G}$. In particular, $\operatorname{Aut}\mathscr{G}$ acts on $\operatorname{span}\{u^1,u^2,u^3,u^4\}$. Let $K=\operatorname{Aut}D$ and H be the group of coordinate automorphisms. Since $u^i\times u^j=2\delta_{ij}u^i$, $u^i/2$, $i=1,2,3,4$ are the primitive idempotents in $\operatorname{span}\{u^1,u^2,u^3,u^4\}$ and thus every automorphism of $\mathscr{G}$ permutes them.

Since K induces S_4 on the set $\{u^1, u^2, u^3, u^4\}$, it follows that Aut $\mathcal{G}$ acts on the set as S_4.

Let C be the kernel of the action of Aut $\mathcal{G}$ on the set and take two linearly independent codewords δ and γ. The product $u^i \times x^\delta$ implies that span$\{x^\delta, x^{-\delta}\}$ is invariant under C. Moreover, the product $x^{\pm\delta} \times x^{\pm\delta}$ and the action of H and K imply that C acts on span$\{x^\delta, x^{-\delta}\}$ as a dihedral group of order 6. The same holds for span$\{x^\gamma, x^{-\gamma}\}$. Since $x^\delta \times x^\gamma = 6x^{\delta+\gamma}$ and since span$\{x^{\delta+\gamma}, x^{-(\delta+\gamma)}\}$ does not contain $x^{\delta-\gamma}$, we conclude that C acts on span$\{x^\delta; 0 \neq \delta \in D\}$ as $(\mathbb{Z}_3 \times \mathbb{Z}_3).2$ and thus Aut $\mathcal{G} \cong 3^2.2S_4$.

Let $G = \operatorname{Aut} M_D$ and take $g \in C_G(\mathcal{G})$. Then g commutes with u^i_1 and u^i_2; $1 \le i \le 4$. Now $\{v \in (M_D)_{(3)} | u^i_2 v = 0 \text{ for } 1 \le i \le 4\}$ is spanned by $q \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$, $\mathbf{1} \otimes q \otimes \mathbf{1} \otimes \mathbf{1}$, $\mathbf{1} \otimes \mathbf{1} \otimes q \otimes \mathbf{1}$, and $\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes q$. Moreover, the eigenvectors for u^1_1 with eigenvalue 3 in this subspace are scalar multiples of $q \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$. Hence g maps $q \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$ to its scalar multiple. If $\delta = (d_1, d_2, d_3, d_4) \in D$ with $d_1 \neq 0$, then $(q \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1})_2 x^\delta$ is a nonzero scalar multiple of x^δ. Since g fixes x^δ, this implies that g fixes $q \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$ also. Likewise, we see that g fixes $\mathbf{1} \otimes q \otimes \mathbf{1} \otimes \mathbf{1}$, $\mathbf{1} \otimes \mathbf{1} \otimes q \otimes \mathbf{1}$, and $\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes q$. Since these elements and $\mathcal{G}$ generate M_D, we have $g = 1$ and the assertion holds. ∎

5. ZHU'S ALGEBRA OF $L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$

Let $(U, Y, \mathbf{1}, \omega)$ be a vertex operator algebra such that $U = U^0 \oplus U^1$, U^0 is a subalgebra isomorphic to $L(\frac{4}{5}, 0)$ with the same Virasoro element ω, and U^1 is a U^0-module isomorphic to $L(\frac{4}{5}, 3)$.

LEMMA 5.1. *One of the following two cases occurs.*

(1) $Y(u, z)v = 0$ *for all* $u, v \in U^1$, U^1 *is an ideal of the vertex operator algebra* (U, Y), *and the automorphism group* Aut U *is isomorphic to the multiplicative group* $\mathbb{C}^\times$.

(2) $Y(u, z)v \neq 0$ *for any nonzero* $u, v \in U^1$, (U, Y) *is a simple vertex operator algebra, and* Aut $U = \langle \rho \rangle$ *is of order two, where* $\rho u = u$ *if* $u \in U^0$ *and* $\rho u = -u$ *if* $u \in U^1$.

Proof. By our hypothesis $Y(u, z)v$ for $u, v \in U^0$ and $u \in U^0$, $v \in U^1$ are given. Moreover, $Y(u, z)v$ for $u \in U^1$, $v \in U^0$ is determined by the skew symmetricity $Y(v, z)u = e^{zL(-1)}Y(u, -z)v$. Let $\pi^i \colon U \to U^i$ be the projection. Then $\pi^i Y(\cdot, z)|_{U^1}$ induces an intertwining operator of type

$$\begin{pmatrix} & U^i & \\ U^1 & & U^1 \end{pmatrix}$$

for U^0-modules. Hence the fusion rule $L(\frac{4}{5},3) \times L(\frac{4}{5},3) = L(\frac{4}{5},0)$ implies that $Y(u,z)v \in U^0[[z,z^{-1}]]$ for $u,v \in U^1$ and that for a fixed nonzero intertwining operator $I(\cdot,z)$ of type

$$\begin{pmatrix} & U^0 & \\ U^1 & & U^1 \end{pmatrix},$$

there is $\lambda \in \mathbb{C}$ such that $Y(u,z)v = \lambda I(u,z)v$ for all $u,v \in U^1$. If $\lambda = 0$, then (1) holds. If $\lambda \neq 0$, then all such vertex operator algebras (U,Y) are isomorphic to each other.

Denote by Vir the Virasoro algebra spanned by $L(n) = \omega_{n+1}$; $n \in \mathbb{Z}$ and 1. Let q be a nonzero highest weight vector of U^1. Then as Vir-modules, U^0 is generated by $\mathbf{1}$ and U^1 is generated by q. Let $\rho \in \operatorname{Aut} U$. Then ρ commutes with the action of Vir, and so ρ is 1 on U^0 and $\rho q = \eta q$ for some $\eta \neq 0$. If $\lambda = 0$, then we can choose η arbitrarily. Suppose $\lambda \neq 0$. In this case $Y(u,z)v \neq 0$ for any nonzero $u,v \in U^1$ by [DL, Proposition 11.9]. Since

$$\rho Y(q,z)q = Y(\rho q,z)\rho q = \eta^2 Y(q,z)q,$$

we have $\eta = \pm 1$ and (2) holds. ∎

From now on we assume that (U,Y) is of case (2) of the above lemma. Explicitly, we shall take (M^0,Y) of Section 4 as (U,Y) and ω^3 as the Virasoro element ω.

We shall use basic properties of Zhu's algebras, which can be found in [DMZ, FZ, W, Z1, Z2]. We also follow the notation in these references. By the definition of $O(U^0)$ and $O(U^1)$ of the vertex operator algebra U^0 and its module U^1, we see that $O(U)$ contains both of them. Zhu's algebra $A(U^0) = U^0/O(U^0)$ is a homomorphic image of the polynomial algebra $\mathbb{C}[x]$ of one variable x;

$$\mathbb{C}[x] \Big/ \Big(\prod_h (x-h) \Big) \cong A(U^0); \qquad x^n \mapsto [\omega]^n,$$

where h runs over $\{0, \frac{1}{40}, \frac{1}{15}, \frac{1}{8}, \frac{2}{5}, \frac{21}{40}, \frac{2}{3}, \frac{7}{5}, \frac{13}{8}, 3\}$. Also, as an $A(U^0)$-bimodule, $A(U^1)$ is a homomorphic image of $\mathbb{C}[x,y]$ and the left action and the right action of $A(U^0)$ are given by

$$x^m y^n \mapsto [\omega]^m * [q] * *[\omega]^n,$$

where q is a highest weight vector of U^1. Now, since $U = U^0 \oplus U^1$ is a vertex operator algebra, [Z2, Lemma 2.1.3] implies that the left and the right action of $A(U^0)$ on $A(U^1)$ are identical modulo $O(U)$. Since $[\omega]$ is in the center of the Zhu's algebra $A(U)$, we have

LEMMA 5.2. *$A(U)$ is commutative and it is a homomorphic image of*

$$\mathbb{C}[x]\Big/\Big(\prod_h (x-h)\Big) \oplus \mathbb{C}[x]\Big/\Big(\prod_h (x-h)\Big).$$

Let (W, Y^W) be a U-module. Decompose W into a direct sum of irreducible U^0-submodules; $W = \oplus_{i\in\Lambda} W^i$ with W^i being irreducible U^0-modules. Let $\pi^j\colon W \to W^j$ be the projection. Then

$$\pi^j Y^W(\cdot, z)|_{W^i}\colon U^1 \to \mathrm{Hom}(W^i, W^j)[[z, z^{-1}]] \tag{5.1}$$

is an intertwining operator of type

$$\begin{pmatrix} & W^j & \\ U^1 & & W^i \end{pmatrix}.$$

For $h = 0, \frac{1}{40}, \frac{1}{15}, \frac{1}{8}, \frac{2}{5}, \frac{21}{40}, \frac{2}{3}$, or $\frac{13}{8}$, consider the subspace $W(h)$ spanned by the homogeneous elements of W whose weights are of the form $h+n$ with $n \in \mathbb{Z}_{\geq 0}$. Then $W = \oplus_h W(h)$ and each $W(h)$ is a U-submodule.

We want to show that $W(h) = 0$ for $h = \frac{1}{40}, \frac{21}{40}, \frac{1}{8}$, or $\frac{13}{8}$. Indeed, $W(\frac{1}{40})$ is the direct sum of all W^i isomorphic to $L(\frac{4}{5}, \frac{1}{40})$. Hence the fusion rule $L(\frac{4}{5}, 3) \times L(\frac{4}{5}, \frac{1}{40}) = L(\frac{4}{5}, \frac{21}{40})$ and (5.1) imply that $Y^W(v, z)W(\frac{1}{40}) = 0$ for all $v \in U^1$. Let S be the set of $v \in U$ such that $Y^W(v, z)W(\frac{1}{40}) = 0$. Then for $u \in U$, $v \in S$, and $w \in W(\frac{1}{40})$, the left hand side of

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y^W(u, z_1)Y^W(v, z_2)w$$

$$-z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)Y^W(v, z_2)Y^W(u, z_1)w$$

$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y^W\big(Y(u, z_0)v, z_2\big)w$$

is 0, and so S is an ideal of the vertex operator algebra (U, Y). Since (U, Y) is simple, we conclude that $W(\frac{1}{40}) = 0$. Similarly, $W(h) = 0$ for $h = \frac{21}{40}, \frac{1}{8}$, or $\frac{13}{8}$.

Now suppose W is an irreducible U-module. Then $W = W(h)$ for some $h = 0, \frac{1}{15}, \frac{2}{5}$, or $\frac{2}{3}$. By Zhu's theory [Z2, Theorem 2.2.2] the top level of W, that is, the homogeneous subspace of the smallest weight, is an irreducible $A(U)$-module. Since $A(U)$ is a commutative algebra, the top level of W

must be of dimension one. Hence if $h = \frac{1}{15}$ or $\frac{2}{3}$, we have $W \cong L(\frac{4}{5}, h)$ as U^0-modules.

If $h = \frac{2}{5}$, the fusion rule, (5.1), and [DL, Proposition 11.9] imply that W^i is isomorphic to $L(\frac{4}{5}, \frac{2}{5})$ or $L(\frac{4}{5}, \frac{7}{5})$ and that there are at least one W^i isomorphic to $L(\frac{4}{5}, \frac{2}{5})$ and at least one W^i isomorphic to $L(\frac{4}{5}, \frac{7}{5})$. Since $A(U)$ acts irreducibly on the top level of W, there is exactly one W^i isomorphic to $L(\frac{4}{5}, \frac{2}{5})$. Suppose there is more than one W^i isomorphic to $L(\frac{4}{5}, \frac{7}{5})$. We may assume that $W^0 \cong L(\frac{4}{5}, \frac{2}{5})$ and $W^1 \cong W^2 \cong L(\frac{4}{5}, \frac{7}{5})$. Let $\psi\colon W^1 \to W^2$ be an isomorphism for U^0-modules. Then

$$Y^W(\cdot, z)|_{W^2}\psi\colon U^1 \to \operatorname{Hom}(W^1, W^0)[[z, z^{-1}]]$$

is an intertwining operator of type

$$\begin{pmatrix} & W^0 & \\ U^1 & & W^1 \end{pmatrix}.$$

Since the set of all intertwining operators of type

$$\begin{pmatrix} & L(\frac{4}{5}, \frac{2}{5}) & \\ L(\frac{4}{5}, 3) & & L(\frac{4}{5}, \frac{7}{5}) \end{pmatrix}$$

is a vector space of dimension one ([W, Theorem 4.3]), there is $0 \neq \mu \in \mathbb{C}$ such that $Y^W(v, z)w = \mu Y^W(v, z)\psi(w)$ for all $v \in U^1$ and $w \in W^1$. Set $S = \{w - \mu\psi(w) | w \in W^1\}$, which is a U^0-submodule isomorphic to W^1. Since $Y^W(v, z)S = 0$ for $v \in U^1$, S is in fact a U-submodule. This contradicts the assumption that W is an irreducible U-module. Therefore W is isomorphic to $L(\frac{4}{5}, \frac{2}{5}) \oplus L(\frac{4}{5}, \frac{7}{5})$ as a U^0-module. If $W = W(0)$, by a similar argument we have W is isomorphic to $L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$ as a U^0-module.

We have shown that

LEMMA 5.3. *Any irreducible U-module is, as a U^0-module, isomorphic to one of*

$$W(0) \cong L\left(\tfrac{4}{5}, 0\right) \oplus L\left(\tfrac{4}{5}, 3\right), \qquad W\left(\tfrac{2}{5}\right) \cong L\left(\tfrac{4}{5}, \tfrac{2}{5}\right) \oplus L\left(\tfrac{4}{5}, \tfrac{7}{5}\right),$$
$$W\left(\tfrac{1}{15}\right) \cong L\left(\tfrac{4}{5}, \tfrac{1}{15}\right), \qquad W\left(\tfrac{2}{3}\right) \cong L\left(\tfrac{4}{5}, \tfrac{2}{3}\right).$$

We need to show the existence of these modules and determine the isomorphism classes. Let $h_1 = \frac{2}{5}$ and $h_2 = \frac{7}{5}$. Suppose (W, Y^W) and $(W, \tilde{Y}^W)$ are irreducible U-modules such that $W = W^1 \oplus W^2$ with W^i being isomorphic to $L(\frac{4}{5}, h_i)$ as a U^0-module for $i = 1, 2$ both in (W, Y^W) and in $(W, \tilde{Y}^W)$. Then there are constants ζ, η such that $\tilde{Y}^W(u, z)w^1 = \zeta Y^W(u, z)w^1$ and $\tilde{Y}^W(u, z)w^2 = \eta Y^W(u, z)w^2$ for all $u \in U^0$, $w^i \in W^i$. Moreover, the fusion rule $L(\frac{4}{5}, 3) \times L(\frac{4}{5}, h_i) = L(\frac{4}{5}, h_j)$ for $\{i, j\} = \{1, 2\}$

implies that there are nonzero constants λ, μ such that $\tilde{Y}^W(v,z)w^1 = \lambda Y^W(v,z)w^1$ and $\tilde{Y}^W(v,z)w^2 = \mu Y^W(v,z)w^2$ for all $v \in U^1$, $w^i \in W^i$. These constants are not independent. Indeed, replace Y^W with $\tilde{Y}^W$ in the identity

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y^W(v,z_1)Y^W(v,z_2)w^i - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)Y^W(v,z_2)Y^W(v,z_1)w^i = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y^W\big(Y(v,z_0)v,z_2\big)w^i,$$

where $v \in U$, $w^i \in W^i$. If $v \in U^0$ and $i = 1$, then the left hand side is multiplied by ζ^2, while the right hand side is multiplied by ζ. Hence by [DL, Proposition 11.9] it follows that $\zeta = 1$. Similarly, $\eta = 1$. If $v \in U^1$, then the left hand side is multiplied by $\lambda\mu$, while the right hand side remains invariant. Thus $\lambda\mu = 1$. Then $(W, \tilde{Y}^W)$ and (W, Y^W) are equivalent U-modules under the isomorphism $w^1 + w^2 \mapsto w^1 + \lambda w^2$. Therefore all irreducible U-modules of type $W(\frac{2}{5})$ of Lemma 5.3 are isomorphic to each other. The existence of such an irreducible U-module follows from Lemma 4.1.

Of course U itself is an irreducible U-module which is isomorphic to $L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$ as a U^0-module. By a similar argument as above we have the uniqueness of such an irreducible U-module.

Suppose next that (W, Y^W) and $(W, \tilde{Y}^W)$ are irreducible U-modules such that W is isomorphic to $L(\frac{4}{5}, \frac{1}{15})$ as a U^0-module both in (W, Y^W) and in $(W, \tilde{Y}^W)$. Then there is a nonzero constant ζ such that $\tilde{Y}^W(u,z) = \zeta Y^W(u,z)$ for all $u \in U^0$. Moreover, the fusion rule $L(\frac{4}{5}, 3) \times L(\frac{4}{5}, \frac{1}{15}) = L(\frac{4}{5}, \frac{1}{15})$ implies that there is a nonzero constant λ such that $\tilde{Y}^W(v,z) = \lambda Y^W(v,z)$ for all $v \in U^1$. Now, by a similar argument as above we have $\zeta = 1$ and $\lambda = \pm 1$. Hence there are at most two isomorphism classes of irreducible U-modules of type $W(\frac{1}{15})$. On the other hand, since $\rho q = -q$ and $\rho(V^1) = V^2$, Lemma 4.2 shows that there are two nonisomorphic U-modules which are, as U^0-modules, isomorphic to $L(\frac{4}{5}, \frac{1}{15})$. As for $L(\frac{4}{5}, \frac{2}{3})$, apply a similar argument.

We have classified the irreducible modules for U. Namely

THEOREM 5.4. *The simple vertex operator algebra U has exactly six isomorphism classes of irreducible modules, which are represented by*

$$W(0), \quad W\big(\tfrac{2}{5}\big), \quad W\big(\tfrac{1}{15}, +\big), \quad W\big(\tfrac{1}{15}, -\big),$$
$$W\big(\tfrac{2}{3}, +\big), \quad W\big(\tfrac{2}{3}, -\big).$$

These irreducible modules are, as U^0-modules, isomorphic to

$$W(0) \cong L\left(\tfrac{4}{5},0\right) \oplus L\left(\tfrac{4}{5},3\right), \qquad W\left(\tfrac{2}{5}\right) \cong L\left(\tfrac{4}{5},\tfrac{2}{5}\right) \oplus L\left(\tfrac{4}{5},\tfrac{7}{5}\right),$$
$$W\left(\tfrac{1}{15},+\right) \cong W\left(\tfrac{1}{15},-\right) \cong L\left(\tfrac{4}{5},\tfrac{1}{15}\right),$$
$$W\left(\tfrac{2}{3},+\right) \cong W\left(\tfrac{2}{3},-\right) \cong L\left(\tfrac{4}{5},\tfrac{2}{3}\right).$$

THEOREM 5.5. *Zhu's algebra $A(U)$ of the vertex operator algebra U is isomorphic to*

$$\mathbb{C}[x]/\left(x\left(x-\tfrac{2}{5}\right)\left(x-\tfrac{1}{15}\right)\left(x-\tfrac{2}{3}\right)\right) \oplus \mathbb{C}[x]/\left(\left(x-\tfrac{1}{15}\right)\left(x-\tfrac{2}{3}\right)\right).$$

More precisely, the image $U^0+O(U)/O(U)$ of U^0 in $A(U)$ is isomorphic to $\mathbb{C}[x]/(x(x-\frac{2}{5})(x-\frac{1}{15})(x-\frac{2}{3}))$ and the image $U^1+O(U)/O(U)$ of U^1 in $A(U)$ is isomorphic to $\mathbb{C}[x]/((x-\frac{1}{15})(x-\frac{2}{3}))$.

Proof. Since the isomorphism classes of irreducible modules for U and the isomorphism classes of the irreducible modules for Zhu's algebra $A(U)$ are in one-to-one correspondence [Z2, Theorem 2.2.2], we can determine $A(U)$ by Lemma 5.2 and Theorem 5.4. Indeed, if $\varphi\colon A(U) \to \operatorname{End} N$ is a representation of the associative algebra, then there exists a module M for U such that $M_{(h)}$ of the smallest weight h of M is equal to N with the action given as follows: $\varphi([v])w = o(v)w$ for $w \in N$ and $v \in U$, where $[v]$ denotes the image of v in $A(U) = U/O(U)$ and $o(v) = v_{\mathrm{wt}\, v-1}$ is the coefficient of $z^{-\mathrm{wt}\, v}$ in $Y(v,z)$.

Let W be an irreducible U-module. Then we know that the smallest weight h of W is one of 0, $\frac{2}{5}$, $\frac{1}{15}$, or $\frac{2}{3}$. Now look at W as a U^0-module. The action of $A(U^0)$ on $W_{(h)}$ is so that $[\omega]$ acts as the multiplication by h ([W, Proposition 4.2]). Hence the image of U^0 in $A(U)$ is isomorphic to $\mathbb{C}[x]/(x(x-\frac{2}{5})(x-\frac{1}{15})(x-\frac{2}{3}))$.

Since the image of U^1 in $A(U)$ is generated by $[q]$ as a $U^0+O(U)/O(U)$-module, it is a homomorphic image of $U^0+O(U)/O(U)$. Consider the case $W = W(\frac{2}{5}) = W^1 \oplus W^2$ with $W^1 \cong L(\frac{4}{5},\frac{2}{5})$ and $W^2 \cong L(\frac{4}{5},\frac{7}{5})$. By the fusion rule, we have $Y^W(v,z)W^1 \subset W^2[[z,z^{-1}]]$ for $v \in U^1$. Since the smallest weights of W^1 and W^2 are different and since $o(v)$ preserves the weights, this implies that $o(v)w = 0$ for all homogeneous elements $w \in W^1$ of weight $\frac{2}{5}$. Hence there is no factor isomorphic to $\mathbb{C}[x]/(x-\frac{2}{5})$ in $U^1+O(U)/O(U)$. Similarly, there is no factor isomorphic to $\mathbb{C}[x]/(x)$.

In the case where $h = \frac{1}{15}$ or $\frac{2}{3}$, the two irreducible U-modules $W(h,+)$ and $W(h,-)$ are isomorphic as U^0-modules, while U^1 acts in different ways. Therefore the factor isomorphic to $\mathbb{C}[x]/(x-h)$ must appear in $U^1+O(U)/O(U)$. ∎

THEOREM 5.6. *The vertex operator algebra U is rational.*

Proof. It is sufficient to show that any U-module W is completely reducible. By the argument before Lemma 5.3, we may assume that the weights of W are of the form $h+n$ with $n \in \mathbb{Z}_{\geq 0}$ and h is one of 0, $\frac{2}{5}$, $\frac{1}{15}$, or $\frac{2}{3}$. Decompose W into a direct sum of irreducible U^0-submodules; $W = \oplus_{i\in\Lambda} W^i$. Then the homogeneous subspace $W^i_{(h)}$ of each irreducible U^0-submodule W^i of weight h is 0 or of dimension one. Denote by Λ_0 the set of all $i \in \Lambda$ such that $W^i_{(h)} \neq 0$. Then $W_{(h)} = \oplus_{i\in\Lambda_0} W^i_{(h)}$. Zhu's algebra $A(U)$ acts on $W_{(h)}$ via $[v]w = o(v)w$ for $v \in U$ and $w \in W_{(h)}$. If $h = 0$ or $\frac{2}{5}$, then as in the proof of Theorem 5.5 the image of U^1 in $A(U)$ acts as zero on each $W^i_{(h)}$, and $W^i_{(h)}$ is in fact invariant under $A(U)$. If $h = \frac{1}{15}$ or $\frac{2}{3}$, then all W^i are isomorphic to $L(\frac{4}{5}, h)$ as U^0-modules and so any nonzero linear combination of $W^i_{(h)}$; $i \in \Lambda_0$ generates a U^0-submodule isomorphic to $L(\frac{4}{5}, h)$. Since eigenvectors of the action of $A(U)$ on $W_{(h)}$ are such linear combinations, we can choose direct summands W^i so that all $W^i_{(h)}$ are invariant under $A(U)$. Let N^i be the U-submodule generated by $W^i_{(h)}$. Then, since $W^i_{(h)}$ is an $A(U)$-submodule, the homogeneous subspace $N^i_{(h)}$ of weight h is equal to $W^i_{(h)}$ by [Z2, Theorem 2.2.1].

We want to show that $W = \oplus_{i\in\Lambda_0} N^i$ and that each N^i is an irreducible U-module. Assume that $h = \frac{1}{15}$ or $\frac{2}{3}$. Then each W^i is isomorphic to $L(\frac{4}{5}, h)$ as a U^0-module. Now N^i can be decomposed into a direct sum of irreducible U^0-submodules. Since $N^i_{(h)} = W^i_{(h)}$, we conclude that $\Lambda_0 = \Lambda$ and $N^i = W^i$. Assume next that $h = 0$ or $\frac{2}{5}$ and set $h' = 3$ or $\frac{7}{5}$, respectively. Consider the decomposition of N^i into a direct sum of irreducible U^0-submodules. Since $N^i_{(h)} = W^i_{(h)}$, there is only one direct summand isomorphic to $L(\frac{4}{5}, h)$, which is of course W^i, and the other direct summands, say M^j; $j \in \Gamma$, are isomorphic to $L(\frac{4}{5}, h')$. We slightly modify the argument before Lemma 5.3. Take a direct summand M^j. If $Y^W(v,z)M^j = 0$ for all $v \in U^1$, then M^j is a submodule for U and U^1 is equal to $\{u \in U | Y^W(u,z)M^j = 0\}$. But then U^1 is an ideal of U, a contradiction. Thus

$$Y^W(\cdot, z)|_{M^j} \colon U^1 \to \mathrm{Hom}(M^j, W^i)[[z, z^{-1}]]$$

is a nonzero intertwining operator of type

$$\begin{pmatrix} & W^i & \\ U^1 & & M^j \end{pmatrix}.$$

Suppose there is another direct summand M^k. Let $\psi\colon M^j \to M^k$ be an isomorphism for U^0-modules. We can view $Y^W(\cdot, z)|_{M^j}$ and $Y^W(\cdot, z)|_{M^k}\psi$

as nonzero intertwining operators of type

$$\begin{pmatrix} & W^i & \\ U^1 & & M^j \end{pmatrix}$$

so that there is a nonzero constant μ satisfying $Y^W(v,z)w = \mu Y^W(v, z)\psi(w)$ for all $v \in U^1$ and $w \in M^j$. Now $S = \{w - \mu\psi(w) | w \in M^j\}$ is a U^0-submodule isomorphic to $L(\frac{4}{5}, h')$, and in fact it is a U-submodule since $Y^W(v,z)S = 0$ for $v \in U^1$. But then $U^1 = \{u \in U | Y^W(u,z)S = 0\}$ is an ideal of U, a contradiction. Therefore $N^i \cong L(\frac{4}{5}, h) \oplus L(\frac{4}{5}, h')$ as U^0-modules, and in particular N^i is an irreducible U-module. Since $W_{(h)} = \oplus_{i \in \Lambda_0} N^i_{(h)}$, we have $W = \oplus_{i \in \Lambda_0} N^i$ as required. ∎

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